

On exponential growth of degrees

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Abstract

A short proof to a recent theorem of Giambruno and Mishchenko is given in this note.

1 The theorem

The following theorem was recently proved by Giambruno and Mishchenko.

Theorem 1.1. *[1, Theorem 1] For every $0 < \alpha < 1$, there exist $\beta > 1$ and $n_0 \in \mathbb{N}$, such that for every partition λ of $n \geq n_0$ with $\max\{\lambda_1, \lambda'_1\} < \alpha n$*

$$f^\lambda \geq \beta^n.$$

The proof of Giambruno and Mishchenko is rather complicated and applies a clever order on the cells of the Young diagram. It should be noted that Theorem 1.1 is an immediate consequence of Rasala's lower bounds on minimal degrees [2, Theorems F and H]. The proof of Rasala is very different and not less complicated; it relies heavily on his theory of degree polynomials. In this short note we suggest a short and relatively simple proof to Theorem 1.1.

First, note that the following weak version is an immediate consequence of the hook-length formula.

Lemma 1.2. *The theorem holds for every $0 < \alpha < \frac{1}{2e}$.*

Proof. Under the assumption, for every $(i, j) \in [\lambda]$

$$h_{i,j} \leq h_{1,1} \leq \lambda_1 + \lambda'_1 \leq 2\alpha n.$$

Hence, by the hook formula together with Stirling formula, for sufficiently large n

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in [\lambda]} h_{i,j}} \geq \frac{n!}{(2\alpha n)^n} \geq \frac{\left(\frac{n}{e}\right)^n}{(2\alpha n)^n} = \beta^n,$$

where, by assumption, $\beta := \frac{1}{2e\alpha} > 1$. □

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2 Two lemmas

Lemma 2.1. *For every $\lambda \vdash n$*

$$\prod_{\substack{(i,j) \in [\lambda] \\ 1 < i}} h_{ij} \leq (n - \lambda_1)!$$

Proof. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t) \vdash n$ let $\bar{\lambda} := (\lambda_2, \dots, \lambda_t) \vdash n - \lambda_1$. Then

$$1 \leq f^{\bar{\lambda}} = \frac{(n - \lambda_1)!}{\prod_{\substack{(i,j) \in [\lambda] \\ 1 < i}} h_{ij}}.$$

□

Lemma 2.2. *For every $\lambda \vdash n$ and $1 \leq k \leq \lambda_1$*

$$\prod_{(1,j) \in [\lambda]} h_{1j} \leq \binom{n}{k} (\lambda_1 + \lfloor n - \frac{\lambda_1}{k} \rfloor)!$$

Proof. Obviously, $h_{1,1} > h_{1,2} > \dots > h_{1,\lambda_1}$. Since $h_{1,1} \leq n$ it follows that

$$\prod_{\substack{(1,j) \in [\lambda] \\ j \leq k}} h_{1j} \leq (n)_k$$

and

$$\prod_{\substack{(1,j) \in [\lambda] \\ k < j}} h_{1j} \leq (h_{1,k})_{\lambda_1 - k}.$$

To complete the proof, notice, that by definition, $\sum_{i=1}^k \bar{\lambda}'_i \leq n - \lambda_1$. Hence $\bar{\lambda}'_k \leq \lfloor \frac{n - \lambda_1}{k} \rfloor$ and thus

$$h_{1,k} = \lambda_1 - k + \lambda'_k = \lambda_1 - k + 1 + \bar{\lambda}'_k \leq \lambda_1 + \bar{\lambda}'_k \leq \lambda_1 + \lfloor \frac{n - \lambda_1}{k} \rfloor.$$

We conclude that

$$\prod_{\substack{(1,j) \in [\lambda] \\ k < j}} h_{1j} \leq (h_{1,k})_{\lambda_1 - k} \leq (\lambda_1 + \lfloor \frac{n - \lambda_1}{k} \rfloor)_{\lambda_1 - k} = \frac{(\lambda_1 + \lfloor \frac{n - \lambda_1}{k} \rfloor)!}{(\lfloor \frac{n - \lambda_1}{k} \rfloor + k)!} \leq \frac{(\lambda_1 + \lfloor \frac{n - \lambda_1}{k} \rfloor)!}{k!}.$$

Thus

$$\prod_{(1,j) \in [\lambda]} h_{1j} = \prod_{\substack{(1,j) \in [\lambda] \\ j \leq k}} h_{1j} \prod_{\substack{(1,j) \in [\lambda] \\ k < j}} h_{1j} \leq (n)_k \frac{\lambda_1 + \lfloor \frac{n - \lambda_1}{k} \rfloor!}{k!} = \binom{n}{k} (\lambda_1 + \lfloor \frac{n - \lambda_1}{k} \rfloor)!.$$

□

3 Proof of Theorem 1.1

For the sake of simplicity the floor notation is omitted in this section.

By Lemmas 2.1 and 2.2,

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in [\lambda]} h_{ij}} = \frac{n!}{\prod_{(1,j) \in [\lambda]} h_{1j} \prod_{\substack{(i,j) \in [\lambda] \\ 1 < i}} h_{ij}} \geq \frac{n!}{(n - \lambda_1)! \binom{n}{k} (\lambda_1 + n - \frac{\lambda_1}{k})!} = \frac{(n - k)! k!}{(n - \lambda_1)! (\lambda_1 + \frac{n - \lambda_1}{k})!}.$$

Denote $\gamma_n := \frac{\lambda_1}{n}$. By Lemma 1.2, we may assume that $\frac{1}{2e} < \gamma_n < \alpha$. Choose $k = \epsilon n$ for a constant $\epsilon = \epsilon(\alpha)$ to be defined later. By the Stirling formula, the lower bound in the RHS asymptotically equals to

$$\frac{((1 - \epsilon)n)! (\epsilon n)!}{((1 - \gamma_n)n)! (\gamma_n n + \frac{1 - \gamma_n}{\epsilon})!} \sim \sqrt{\frac{\epsilon(1 - \epsilon)}{(1 - \gamma_n)(\gamma_n + \frac{\epsilon n}{n})}} \cdot \frac{(1 - \epsilon)^{(1 - \epsilon)n} \epsilon^{\epsilon n}}{(1 - \gamma_n)^{(1 - \gamma_n)n} (\gamma_n + \frac{\epsilon n}{n})^{(\gamma_n + \frac{\epsilon n}{n})n}} \cdot \left(\frac{e}{n}\right)^{c_n},$$

where $c_n := \frac{1 - \gamma_n}{\epsilon}$. Thus $\frac{1 - \alpha}{\epsilon} < c_n < \frac{2e - 1}{2e\epsilon}$. Hence, for sufficiently large n

$$\begin{aligned} \liminf_{n \rightarrow \infty} (f^\lambda)^{1/n} &\geq \liminf_{n \rightarrow \infty} \left(\sqrt{\frac{\epsilon(1 - \epsilon)}{(1 - \gamma_n)(\gamma_n + \frac{\epsilon n}{n})}} \cdot \frac{(1 - \epsilon)^{(1 - \epsilon)n} \epsilon^{\epsilon n}}{(1 - \gamma_n)^{(1 - \gamma_n)n} (\gamma_n + \frac{\epsilon n}{n})^{(\gamma_n + \frac{\epsilon n}{n})n}} \cdot \left(\frac{e}{n}\right)^{c_n} \right)^{1/n} \\ &\geq \min_{\gamma \in [\frac{1}{2e}, \alpha]} \frac{\epsilon^\epsilon (1 - \epsilon)^{1 - \epsilon}}{\gamma^\gamma (1 - \gamma)^{1 - \gamma}}. \end{aligned}$$

The function $f(x) := x^x (1 - x)^{1 - x}$ is differentiable in the open interval $(0, 1)$, symmetric around its minimum at $x = \frac{1}{2}$, decreasing in $(0, \frac{1}{2}]$, increasing in $[\frac{1}{2}, 1)$, strictly less than 1 in this interval and tends to 1 at the boundaries. Since $\gamma_n \in [\frac{1}{2e}, \alpha] \subseteq (0, 1)$ it follows that $f(\gamma_n) \leq f(\alpha)$ if $1 - \frac{1}{2e} \leq \alpha$ and $\leq f(\frac{1}{2e})$ otherwise. Choosing $\epsilon = \epsilon(\alpha)$ such that $\epsilon \leq \delta \min\{1 - \alpha, \frac{1}{2e}\}$ for some very small $\delta > 0$ we conclude that

$$\liminf_{n \rightarrow \infty} (f^\lambda)^{1/n} \geq \min_{\gamma \in [\frac{1}{2e}, \alpha]} \frac{\epsilon^\epsilon (1 - \epsilon)^{1 - \epsilon}}{\gamma^\gamma (1 - \gamma)^{1 - \gamma}} \geq \min\left\{ \frac{f(\delta \frac{1}{2e})}{f(\frac{1}{2e})}, \frac{f(\delta(1 - \alpha))}{f(1 - \alpha)} \right\} > 1,$$

completing the proof. □

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References

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- [2] R. Rasala, *On the minimal degrees of characters of S_n* , J. Algebra **45** (1977), 132-181